I.3 Boundary identification of two manifolds M_1, M_2 with $f : \partial M_1 \xrightarrow{\cong} \partial M_2$

Fact Let M be a topological (or $\mathcal{C}^\infty$) manifold with $\partial M \neq \emptyset$. Then there exists an open neighborhood U of ∂M such that $U \underset{h}{\overset{\cong}{\leftarrow}} \partial M \times [0,1)$ with $h(x,0) = x,\ x \in \partial M$.
U is called a collar.

The idea of proof
Use local collar and splice together using Partition of Unity.
(See Vick(homology theory) for topological case and Milnor(h-cobordism) for $\mathcal{C}^\infty$ case.)

Let $(X, A) = (M_1, \partial M_1)$ (a collared pair), $Y = M_2$ and let $f : \partial M_1 \to \partial M_2$ be a homeomorphism. Then we obtain a manifold $W = M_1 \bigcup_f M_2$ identifying the boundaries of M_1 and M_2 via f.

Examples
1.(Sphere)
Let $f : S^{n-1} \to S^{n-1}$ be a homeomorphism.
Then $W = D^n \bigcup_f D^n \cong S^n$(homeomorphism). But, not diffeomorphism, in general, for $\mathcal{C}^\infty$-category.

증명 Let $\bar{f} : D^n \xrightarrow{\cong} D^n$ be an extension of f.
(Such extension always exists, e.g., a "radial extension".)

$$\begin{array}{ccc} D^n \coprod D^n & \xrightarrow{\bar{f} \coprod id.} & D^n \coprod D^n \\ \downarrow p & & \downarrow p \\ D^n \bigcup_f D^n & \dashrightarrow & D^n \bigcup_{id.} D^n = S^n \end{array} \qquad \begin{array}{ccc} x, f(x) & \longrightarrow & f(x), f(x) \\ \downarrow & & \downarrow \\ x \sim f(x) & & f(x) \sim f(x) \end{array}$$

왼쪽의 그림에서 점선의 map은 오른쪽 그림과 같이 $D^n \bigcup_f D^n$의 $x \sim f(x), x \in \partial D^n (= S^{n-1})$가 $D^n \bigcup_{id.} D^n$의 $f(x) \sim f(x)$로 보내지므로 잘 정의된다. 따라서 자명한 continuous, one-to-one 그리고 onto조건이 더해져서 homeomorphism이 된다.

$\square$

Exactly same argument shows the following.

Theorem A
Given M_i with $\partial M_i \neq \emptyset$, if $f, g : \partial M_1 \xrightarrow{\cong} \partial M_2$ such that $g^{-1} \circ f : \partial M_1 \to \partial M_1$ can be extended to a homeomorphism $M_1 \to M_1$, then $M_1 \bigcup_f M_2 \cong M_1 \bigcup_g M_2$.

증명

$$\begin{array}{ccc} M_1 \coprod M_2 & \xrightarrow{\overline{g^{-1}\circ f} \coprod id} & M_1 \coprod M_2 \\ \downarrow p & & \downarrow p \\ M_1 \bigcup_f M_2 & \overset{\phi}{\dashrightarrow} & M_1 \bigcup_g M_2 \end{array} \qquad \begin{array}{ccc} x, f(x) & \longrightarrow & g^{-1}\circ f(x), f(x) \\ \downarrow & & \downarrow \\ x \sim f(x) & & f(x) \sim f(x) \end{array}$$

왼쪽의 그림에서 점선 ϕ는 오른쪽 그림에서 보듯이 위와 같은 이유로 잘 정의된다. 또한 $f^{-1} \circ g = (g^{-1} \circ f)^{-1}$이므로 extension이 여전히 유효하고 같은 방법으로 $\overline{f^{-1} \circ g} = (\overline{g^{-1} \circ f})^{-1}$인 $\psi = \phi^{-1}$를 얻을 수 있다.

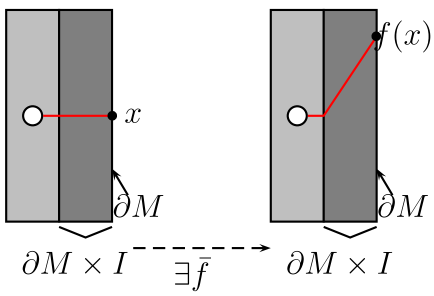
□

2.(Lens space)
Let $M_i = D^2 \times S^1$ with $\partial M_i = S^1 \times S^1 = T^2$. Consider $f : T^2 \xrightarrow{\cong} T^2$ and what $M_1 \bigcup_f M_2$ is.

Theorem B
Let M_i with $\partial M_i \neq \emptyset$ be manifolds and $f, g : \partial M_1 \xrightarrow{\cong} \partial M_2$ be homeomorphisms isotopic to each other. Then $M_1 \bigcup_f M_2 \cong M_1 \bigcup_g M_2$.

증명 By Theorem A, it suffices to show the following statement :
claim $f : \partial M \to \partial M$ isotopic to id.. Then f can be extended to a homeomorphism $\bar{f} : M \to M$.

Proof of claim



such that $\bar{f}(x,t) = (f_t(x), t)$on a collar neighborhood $U \cong M \times [0,1)$ and define $\bar{f} = id.$ on the complement of U.

□

Question : What are homeomorphisms of T^2 up to isotopy?
Answer : Classical result $\Rightarrow Gl(2,\mathbb{Z})$.
In fact, $Homeo(T^2) \overset{\phi}{\twoheadrightarrow} Aut(H_1(T^2))(= Gl(2,\mathbb{Z}))$
$f \mapsto f_*$
with $ker(\phi) = Homeo_0(T^2) =$ homeomorphism homotopic(isotopic) to $id.$.

$\therefore \mathcal{H}(T^2)/\mathcal{H}_0(T^2) \cong Gl(2,\mathbb{Z})$.

Show ϕ is onto ; Given $g \in Gl(2,\mathbb{Z})$, view $g : \mathbb{R}^2 \to \mathbb{R}^2$ preserving the integral lattice. $\Rightarrow \bar{g} : T^2 \to T^2$ is induced.

일반적으로 closed surface 에서 "homotopy $\Rightarrow$ isotopy"가 성립하며 $ker(\phi) = Homeo_0(T^2)$는 다음 exercise 2(3)을 이용하여 $\tilde{f}$와 $id.$사이에 straight line homotopy를 줄 수 있고, 이것을 T^2상에서의 homotopy 로 내릴 수 있다.

HW 7 Prove the followings.

명제 1

$$\begin{array}{ccccc} \widetilde{X} & \overset{\tilde{f}}{\to} & \widetilde{X'} & & \forall \gamma \in \Gamma = \textit{deck group of } \widetilde{X}, \\ \downarrow^p & & \downarrow^p & \Leftrightarrow & \exists! \gamma' \in \Gamma' \textit{such that} \\ X & \overset{\exists f}{\dashrightarrow} & X' & & \widetilde{f} \circ \gamma = \gamma' \circ \widetilde{f} \end{array}$$

where p is a regular covering.

따름정리 2 *(1)*

$$\begin{array}{ccc} \widetilde{X} & \overset{\tilde{f}}{\underset{\cong}{\to}} & \widetilde{X'} \\ \downarrow p & & \downarrow p' \\ X & \overset{\exists f}{\underset{\cong}{\dashrightarrow}} & X' \end{array} \quad \Leftrightarrow \quad C_{\tilde{f}} : \Gamma \overset{\cong}{\to} \Gamma'$$

(2) 위에서 $X = X'$인 경우 좌변 $\Leftrightarrow \widetilde{f} \in N_G(\Gamma)$, when $G = Aut(\widetilde{X})$.(The type of the automorphisms depends on the category.)

(3) Universal covering case의 경우에는
$\widetilde{f} \in C_G(\Gamma)$ *(= Centralizer)* $\Leftrightarrow f_* = id. : \pi_1(X) \to \pi_1(X)$ *when* $\widetilde{f}$ *fixes a base point.*

Q. Suppose ${M_i}^3$ be a 3-manifold with $\partial M_i = T^2, \quad i = 1, 2$.
When is $M_1 \underset{f}{\cup} M_2 \cong M_1 \underset{g}{\cup} M_2$, where $f, g : T^2 \to T^2$ are homeomorphism?

A. Theorem A $\Rightarrow$ If $g^{-1} \cdot f : T^2 \to T^2$ can be extended to a homeomorphism: $M_1 \to M_1$, then $M_1 \underset{f}{\cup} M_2 \cong M_1 \underset{g}{\cup} M_2$.

명제 3 *Suppose* $M_1 = D^2 \times S^1$*, a solid torus. Then* $h : \partial M_1 = T^2 \to T^2$ *can be extended to* $\bar{h} : M_1 \to M_1$ *iff* $h_* : H_1(T^2) = \mathbb{Z} \cdot m \oplus \mathbb{Z} \cdot l \to H_1(T^2)$ *is of the form* $\begin{pmatrix} \pm 1 & c \\ 0 & \pm 1 \end{pmatrix}$*, equivalently, h sends meridian to meridian.*

증명 ($\Rightarrow$)

$$\begin{array}{ccc} H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} = < m, l > & \xrightarrow{h_* : \cong} & \mathbb{Z} \oplus \mathbb{Z} \\ \downarrow i_* & & \downarrow \\ H_1(D^2 \times S^1) = \mathbb{Z} = < l > & \xrightarrow{\bar{h}_*} & \mathbb{Z} \end{array} \qquad \begin{array}{ccc} m & \longmapsto & am + bl \\ \downarrow & & \downarrow \\ 0 & \longmapsto & bl \end{array}$$

$\Rightarrow b = 0$ and $a = \pm 1 (\because h_* : m \mapsto am : \cong)$
$\Rightarrow h_* = \begin{pmatrix} \pm 1 & c \\ 0 & d \end{pmatrix}$ and h_* is invertible.
$\Rightarrow h_* = \begin{pmatrix} \pm 1 & c \\ 0 & \pm 1 \end{pmatrix}$(복호동순 아님)

($\Leftarrow$) May assume h is a "linear map" through isotopy in the radial direction. And it is enough to check for $h = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and each of these is clearly a restriction of homeomorphism of a solid torus. □

따름정리 4 $M_1 = D^2 \times S^1, \partial M_2 = T^2$ *and let* $h : \partial M_1 \xrightarrow{\cong} \partial M_2$.
Then the topological type of $M_1 \underset{h}{\cup} M_2$ *depends only on* $|h_*(m)|$*, i.e., if* $h_*(m) = \pm h'_*(m)$*, then* $M_1 \underset{h}{\cup} M_2 \cong M_1 \underset{h'}{\cup} M_2$.

증명 Suppose $h, h' : \partial M_1 \xrightarrow{\cong} \partial M_2$ s.t. $h_*(m) = \pm h'_*(m)$. Then $(h^{-1}h')_*(m) = \pm m$. Proposition and Theorem A $\Rightarrow M_1 \underset{h}{\cup} M_2 \cong M_1 \underset{h'}{\cup} M_2$. □

Lens Space

$L(p,q) = D^2 \times S^1 \underset{h}{\cup} D^2 \times S^1$, $h(m) = qm + pl$, where p and q are relatively prime.[1]

Note. $L(1,0) = S^3$
$L(0,1) = S^2 \times S^1$
$L(2,1) = \mathbb{R}P^2$

숙제 8. (1) $L(p,q) \cong L(p,-q) \cong L(-p,q) \cong L(-p,-q) \cong L(p,q+kp)$ $\forall k \in \mathbb{Z}$
(2)$\pi_1(L(p,q)) = \mathbb{Z}/p$ (Use Van-Kampen Theorem.)

Homology of $L(p,q)$

$$\cdots \longrightarrow H_q(S^1 \times S^1) \xrightarrow{(i_*,h_*)} H_q(D^2 \times S^1) \oplus H_q(D^2 \times S^1) \longrightarrow H_q(L) \longrightarrow \cdots$$

$\Rightarrow H_q(L) = 0$ if $q \geq 4$.

$$0 \to H_3(L) \to H_2(S^1 \times S^1) \to 0 \to H_2(L) \to H_1(S^1 \times S^1) \to \mathbb{Z} \oplus \mathbb{Z} \to H_1(L) \to \tilde{H}_0(S^1 \times S^1)$$

, where $h_* = \begin{pmatrix} q & r \\ p & s \end{pmatrix} \Rightarrow$
$(i_*, h_*) = \begin{pmatrix} 0 & 1 \\ p & s \end{pmatrix} : H_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z} \to H_1(D^2 \times S^1) \oplus H_1(D^2 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$

(1) $p \neq 0 \Rightarrow det \neq 0$ and (i_*, h_*) is injective.$\Rightarrow$

$$\to H_2(L) \longrightarrow H_1(S^1 \times S^1) \to$$
$$H_2(L) \searrow 0 \nearrow H_1(S^1 \times S^1)$$

$\Rightarrow H_2(L) = 0$ and $H_1(L) = \mathbb{Z} \oplus \mathbb{Z}/im(i_*, h_*) = \mathbb{Z}/p$

[1]Because h_* is homeomorphism, determinant of h_* is ± 1. Then $qs - rp = \pm 1$, i.e., p and q are relatively prime.

(2) $p = 0 \Rightarrow h(m) = qm \overset{h:isomorphism}{\Rightarrow} q = \pm 1$ (meridian$\mapsto$ meridian)

$H_2(L) = ker \begin{pmatrix} 0 & 1 \\ 0 & s \end{pmatrix} = \mathbb{Z}$ and $H_1(L) = cok \begin{pmatrix} 0 & 1 \\ 0 & s \end{pmatrix} = \mathbb{Z}$

In this case, $L = S^2 \times S^1$.

Remark. (1) $p \neq p' \Rightarrow L(p,q) \not\cong L(p',q')$

(2) Fact. $L(p,q) \cong L(p,q') \Leftrightarrow q' = \pm q^{\pm 1} (mod\ p)$

$L(p,q) \simeq L(p,q') \Leftrightarrow qq' = \pm m^2 (mod\ p)$ for some m

(Ref. M. Cohen. A course in simple homotopy theory)